

# EQUIVARIANT SPECTRAL SEQUENCES FOR LOCAL COEFFICIENTS

MEGAN GUICHARD SHULMAN

ABSTRACT. We recall how a description of local coefficients that Eilenberg introduced in the 1940s leads to spectral sequences for the computation of homology and cohomology with local coefficients. We then show how to construct new equivariant analogues of these spectral sequences and give a worked example of how to apply them in a computation involving the equivariant Serre spectral sequence.

This paper contains some of the material in the author's Ph.D. thesis, which also discusses the results of L. Gaunce Lewis [Lew88] on the cohomology of complex projective spaces and corrects some flaws in that paper.

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Applications of the Serre spectral sequence in the literature usually use trivial local coefficients. This perhaps reflects the tendency of much of modern algebraic topology to steer away from the unstable world, where the fundamental group cannot be ignored. But it may also reflect our lack of tools for computing the relevant homology and cohomology with local coefficients. Whatever the reason, it is impossible to ignore local coefficients when working equivariantly with Bredon (co)homology and the equivariant Serre spectral sequence; almost no interesting examples reduce to trivial local coefficients. It is thus necessary to develop some tools for working with homology and cohomology with local coefficients.

To this end, we recall a simple universal coefficient spectral sequence which appears in [CE56, p. 355], but which, to the best of our knowledge, has not previously

been applied in conjunction with the Serre spectral sequence. Part of the point is that the definition of local coefficients that appears in the construction of the Serre spectral sequence is not tautologically the same as the definition that gives the cited universal coefficient spectral sequence.

The connection comes from an old result of Eilenberg [Eil47], popularized by Whitehead [Whi78, VI.3.4] and, more recently, by Hatcher [Hat02, App3.H]. It identifies the local coefficients that appear in the context of fibrations with a more elementary definition in terms of the chains of the universal cover of the base space. The identification makes working with local coefficients much more feasible. We shall first say in Section 1 how this goes nonequivariantly, and illustrate with an example. We then explain the equivariant generalization in Section 2. The later parts of the paper will develop some necessary background and finally go through a sample calculation in Section 7.

## 1. THE NONEQUIVARIANT SITUATION

Let  $X$  be a path-connected based space with universal cover  $\tilde{X}$ . Let  $\pi = \pi_1(X)$  and let  $\pi$  act on the right of  $\tilde{X}$  by deck transformations. Fix a ring  $R$ , and let  $M$  be a left and  $N$  be a right module over the group ring  $R[\pi]$ . Let  $C_*$  be the normalized singular chain complex functor with coefficients in  $R$ .

**Definition 1.1.** Define the homology of  $X$  with coefficients in  $M$  by

$$H_*(X; M) = H_*(C_*(\tilde{X}) \otimes_{R[\pi]} M).$$

Define the cohomology of  $X$  with coefficients in  $N$  by

$$H^*(X; N) = H^*(\text{Hom}_{R[\pi]}(C_*(\tilde{X}), N)).$$

Functoriality in  $M$  and  $N$  for fixed  $X$  is clear. For a based map  $f: X \rightarrow Y$ , where  $\pi_1(Y) = \rho$ , and for a left  $R[\rho]$ -module  $P$ , we may regard  $P$  as a  $R[\pi]$ -module by pullback along  $\pi_1(f)$ , and then, using the standard functorial construction of the universal cover, we obtain

$$f_*: H_*(X; f^*P) \rightarrow H_*(Y; P).$$

Cohomological functoriality is similar. The definition goes back to Eilenberg [Eil47], and has the homology of spaces and the homology of groups as special cases, as discussed below. It deserves more emphasis than it is usually given because it implies spectral sequences for the calculation of homology and cohomology with local coefficients, as we shall recall.

**Example 1.2.** If  $\pi$  acts trivially on  $M$  and  $N$ , then  $H_*(X; M)$  and  $H^*(X; N)$  are the usual homology and cohomology groups of  $X$  with coefficients in  $M$  and  $N$ . We can identify  $C_*(X)$  with  $C_*(\tilde{X}) \otimes_{R[\pi]} R$ , where  $R[\pi]$  acts trivially on  $R$ . This implies the identifications

$$C_*(\tilde{X}) \otimes_{R[\pi]} M \cong C_*(X; M) \quad \text{and} \quad \text{Hom}_{R[\pi]}(C_*(\tilde{X}), N) \cong C^*(X; N).$$

**Example 1.3.** If  $X = K(\pi, 1)$ , then  $H_*(X; M)$  and  $H^*(X; N)$  are the usual homology and cohomology groups of  $\pi$  with coefficients in  $M$  and  $N$  since  $C_*(\tilde{X})$  is an  $R[\pi]$ -free resolution of  $R$ . That is,

$$H_*(K(\pi, 1); M) = \text{Tor}_*^{R[\pi]}(R, M) \quad \text{and} \quad H^*(K(\pi, 1); M) = \text{Ext}_{R[\pi]}^*(R, N).$$

**Example 1.4.** If  $M = R[\pi] \otimes_R A$  and  $N = \text{Hom}_R(R[\pi], A)$  for an  $R$ -module  $A$ , then

$$H_*(X; M) \cong H_*(\tilde{X}, A) \quad \text{and} \quad H^*(X; N) \cong H^*(\tilde{X}; A).$$

*Remark 1.5.* If we replace  $N$  by  $M$  (viewed as a right  $R[\pi]$ -module) in the cohomology case of the previous example, then we are forced to impose finiteness restrictions and consider cohomology with compact supports; compare [Hat02, 3H.5].

We have spectral sequences that generalize the last two examples. When  $\pi$  acts trivially on  $M$  and  $N$ , they can be thought of as versions of the Serre spectral sequence of the evident fibration  $\tilde{X} \rightarrow X \rightarrow K(\pi, 1)$ .

**Theorem 1.6** (Eilenberg Spectral Sequence). *There are spectral sequences*

$$E_{p,q}^2 = \text{Tor}_{p,q}^{R[\pi]}(H_*(\tilde{X}), M) \implies H_{p+q}(X; M)$$

and

$$E_2^{p,q} = \text{Ext}_{R[\pi]}^{p,q}(H_*(\tilde{X}), N) \implies H^{p+q}(X; N).$$

Up to notation, these are the spectral sequences given by Cartan and Eilenberg in [CE56, p. 355].

*Proof.* In the  $E^2$  and  $E_2$  terms,  $p$  is the homological degree and  $q$  is the internal grading on  $H_*(\tilde{X})$ . Let  $\varepsilon: P_* \rightarrow M$  be an  $R[\pi]$ -projective resolution of  $M$  and form the bicomplex

$$C_*(\tilde{X}) \otimes_{R[\pi]} P_*;$$

the theorem comes from looking at the two spectral sequences associated to this bicomplex and converging to a common target.

If we filter  $C_*(\tilde{X}) \otimes_{R[\pi]} P_*$  by the degrees of  $C_*(\tilde{X})$ , we get a spectral sequence whose  $E^0$ -term has differential  $\text{id} \otimes d$ . Since  $C_*(\tilde{X})$  is a projective  $R[\pi]$  module, the resulting  $E^1$ -term is  $C_*(\tilde{X}) \otimes_{R[\pi]} M$ , the resulting  $E^2$ -term is  $H_*(X; M)$ , and  $E^2 = E^\infty$ . Since  $E^\infty$  is concentrated in degree  $q = 0$ , there is no extension problem; we have identified the target as claimed in the theorem.

Filtering the other way, by the degrees of  $P_*$ , we obtain a spectral sequence whose  $E^0$ -term has differential  $d \otimes \text{id}$ . The resulting  $E^1$ -term is  $H_*(\tilde{X}) \otimes_{R[\pi]} P_*$  and the resulting  $E^2$ -term is  $\text{Tor}_{*,*}^{R[\pi]}(H_*(\tilde{X}), M)$ . This gives the first statement of the theorem.

The argument in cohomology is similar, starting from the bicomplex

$$\text{Hom}_{R[\pi]}(C_*(\tilde{X}), I^*)$$

for an injective resolution  $\eta: N \rightarrow I^*$  of  $N$ . □

We record an immediate corollary.

**Corollary 1.7.** *Let  $\pi$  be a finite group of order  $n$  and  $R$  be a field of characteristic prime to  $n$ . Then*

$$H_*(X; M) \cong H_*(\tilde{X}) \otimes_{R[\pi]} M \quad \text{and} \quad H^*(X; N) \cong \text{Hom}_{R[\pi]}(H_*(\tilde{X}), N).$$

*Proof.* Since  $R[\pi]$  is semi-simple,  $E_{p,q}^2 = 0$  and  $E_2^{p,q} = 0$  for  $p > 0$ . Therefore the spectral sequences collapse to the claimed isomorphisms. □

If  $\pi$  acts trivially on  $H_*(\tilde{X})$ , so that  $H_*(\tilde{X}) \cong H_*(\tilde{X}) \otimes_{R[\pi]} R$ , then the situation simplifies even further. For ease of notation, let  $M_\pi$  denote the coinvariants  $M/IM$ , where  $I \subset R[\pi]$  is the augmentation ideal, and let  $N^\pi$  denote the fixed points of  $N$ .

**Corollary 1.8.** *Suppose that we are in the situation of Corollary 1.7 and that  $\pi$  acts trivially on  $H_*(\tilde{X})$ . Then*

$$H_*(X; M) \cong H_*(\tilde{X}; M_\pi) \quad \text{and} \quad H^*(X; N) \cong H^*(\tilde{X}; N^\pi). \quad \square$$

It remains to identify the homology and cohomology groups of Definition 1.1 with classical (co)homology with local coefficients. To do this, we first need to reconcile our coefficient  $R[\pi]$ -modules with the classical definition of a local coefficient system.

In Definition 1.1, we took  $M$  and  $N$  to be left and right modules over the group ring  $R[\pi]$  and took  $C_*(\tilde{X})$  to be the normalized singular chains of  $\tilde{X}$ . A (left or right)  $R[\pi]$ -module  $M$  is the same as a (covariant or contravariant) functor from  $\pi$ , viewed as a category with a single object, to the category of  $R$ -modules.

As usual, given a space  $X$ , let  $\Pi X$  be the fundamental groupoid of  $X$ ; this is a category whose objects are the points of  $X$  and whose morphism sets are homotopy classes of paths between fixed endpoints. By definition, a **local coefficient system**  $\mathcal{M}$  on  $X$  is a functor (covariant or contravariant depending on context, corresponding to our left and right  $R[\pi]$ -module distinction above) from the fundamental groupoid  $\Pi X$  to the category of  $R$ -modules. When  $X$  is path-connected with basepoint  $x_0$ , the category  $\pi = \pi_1(X)$  with single object  $x_0$  is a skeleton of  $\Pi X$ . Therefore a coefficient system  $\mathcal{M}$  is determined by its restriction  $M$  to  $\pi$ .

Whitehead [Whi78, VI.3.4 and 3.4\*] (see also Hatcher [Hat02, 3H.4]) proves the following result and ascribes it to Eilenberg [Eil47].

**Theorem 1.9** (Eilenberg). *For path-connected spaces  $X$  and covariant and contravariant local coefficient systems  $\mathcal{M}$  and  $\mathcal{N}$  on  $X$ , the classical homology and cohomology with local coefficients  $H_*(X; \mathcal{M})$  and  $H^*(X; \mathcal{N})$  are naturally isomorphic to the homology and cohomology groups  $H_*(X; M)$  and  $H^*(X; N)$ , where  $M$  and  $N$  are the restrictions of  $\mathcal{M}$  and  $\mathcal{N}$  to  $\pi$ .*

Therefore Theorem 1.6 gives a way to compute the additive structure of homology and cohomology with local coefficients. In particular, if  $f: E \rightarrow X$  is a fibration with fiber  $F$  and path-connected base space  $X$ , it gives a means to compute the homology and cohomology with local coefficients that appear in

$$E_{*,*}^2 = H_*(X; \mathcal{H}_*(F; R)) \quad \text{and} \quad E_2^{*,*} = H^*(X; \mathcal{H}^*(F; R))$$

of the Serre spectral sequences for the computation of  $H_*(E; R)$  and  $H^*(E; R)$ .

Even the case when  $\pi$  is finite of order  $n$  and  $R$  is a field of characteristic prime to  $n$  often occurs in practice. More generally, the spectral sequences of Theorem 1.6 help make the Serre spectral sequence amenable to explicit calculation in the presence of non-trivial local coefficient systems.

Note that we have not yet addressed the multiplicative structure of the cohomological Eilenberg spectral sequence; this is work in progress and is discussed in the author's Ph.D. thesis. However, even without the multiplicative structure, we can already do one example. It will be useful to have the following simple consequence of Definition 1.1.

**Proposition 1.10.** *Let  $X$  be a space and  $\pi$  its fundamental group. For any  $R[\pi]$ -module  $M$ ,  $H^0(X; M) \cong M^\pi$ , the  $\pi$ -fixed points of  $M$ . If  $M$  is an  $R[\pi]$ -algebra, the isomorphism is as algebras.*

*Proof.* We would like to identify the kernel of

$$\mathrm{Hom}_{R[\pi]}(C_0(\tilde{X}), M) \longrightarrow \mathrm{Hom}_{R[\pi]}(C_1(\tilde{X}), M).$$

Since  $\tilde{X}$  is connected,  $H_0(\tilde{X}) \cong R$  with the trivial  $\pi$  action. Since  $\mathrm{Hom}_{R[\pi]}$  is left exact, it follows that  $H^0(X; M) \cong \mathrm{Hom}_{R[\pi]}(R, M) \cong M^\pi$ , as claimed.  $\square$

In particular, in the Serre spectral sequence,  $E_2^{0,t} \cong \mathcal{H}^t(F; R)^\pi$ . If we are in a situation where  $E_2^{s,t}$  vanishes for  $s > 0$ , then Proposition 1.10 completely describes the multiplicative structure on the  $E_2$  page.

**Example 1.11.** Let  $\mathbb{Z}/2$  be the cyclic group of order two, identified with  $\{\pm 1\}$  when convenient. Consider the fibration

$$B\mathrm{det}: BO(2) \longrightarrow B\mathbb{Z}/2$$

with fiber  $BSO(2)$ . Take coefficients in a finite field  $R = \mathbb{F}_q$  with  $q$  an odd prime, so that  $R[\pi]$  is semisimple. The action of the base on the fiber is nontrivial; the Serre spectral sequence for this fibration has

$$E_2^{s,t} = H^s(B\mathbb{Z}/2; \mathcal{H}^t(BSO(2); \mathbb{F}_q)) \implies H^{s+t}(BO(2); \mathbb{F}_q)$$

We know that  $B\mathbb{Z}/2 \simeq \mathbb{R}P^\infty$  with universal cover  $S^\infty \simeq \bullet$ , so Corollary 1.8 applies, and  $BSO(2) \simeq \mathbb{C}P^\infty$ . We also know  $H^*(\mathbb{C}P^\infty; \mathbb{F}_q) \cong \mathbb{F}_q[x]$ , a polynomial algebra on one generator  $x$  in degree two, and that the fundamental group  $\pi_1(B\mathbb{Z}/2) \cong \mathbb{Z}/2$  acts on  $H^*(\mathbb{C}P^\infty)$  by  $x \mapsto -x$ .

By Corollary 1.8, we thus have

$$E_2^{s,t} \cong H^s\left(S^\infty; \mathcal{H}^t(BSO(2); \mathbb{F}_q)^{\mathbb{Z}/2}\right)$$

in the Serre spectral sequence;  $\mathcal{H}^t(BSO(2); \mathbb{F}_q)^{\mathbb{Z}/2}$  is either 0 or  $\mathbb{F}_q$ , depending on  $t$ .  $E_2^{s,t}$  thus vanishes for  $s > 0$ , so the Serre spectral sequence collapses with no extension problems. Using the observation after Proposition 1.10, we see

$$H^*(BO(2); \mathbb{F}_q) \cong \mathcal{H}^*(BSO(2); \mathbb{F}_q)^{\mathbb{Z}/2} \cong \mathbb{F}_q[x^2].$$

The isomorphism is of  $R$ -algebras. We have thus shown the well-known fact that  $H^*(BO(2); \mathbb{F}_q)$  is polynomial on one generator (the Pontrjagin class) in degree four. We will later examine an equivariant version of this example.

## 2. EQUIVARIANT GENERALIZATIONS

Heading towards an equivariant generalization of Theorem 1.6, we first rephrase the definition of (co)homology with local coefficients. In Section 1, we effectively defined homology and cohomology with local coefficients by restricting a local coefficient system  $\mathcal{M}: \Pi X \longrightarrow \mathbb{k}\text{-mod}$  to an  $R[\pi]$ -module  $M: \pi \longrightarrow \mathbb{k}\text{-mod}$ .

Rather than restricting  $\mathcal{M}$  to  $\pi$ , we could instead redefine  $\tilde{X}$  to be the universal cover functor  $\Pi X^{\mathrm{op}} \longrightarrow \mathrm{Top}$  that sends a point  $x \in X$  to the space  $\tilde{X}(x)$  of equivalence classes of paths starting at  $x$  and sends a path  $\gamma$  from  $x$  to  $y$  to the map  $\tilde{X}(y) \longrightarrow \tilde{X}(x)$  given by precomposition with  $\gamma$ . Since  $\pi$  is a skeleton of  $\Pi X$ , the following definition is equivalent to Definition 1.1 when  $X$  is connected.

By Theorem 1.9, there is no conflict with the classical notation for homology with local coefficients. Let  $\text{Ch}_{\mathbb{k}}$  denote the category of chain complexes of  $R$ -modules.

**Definition 2.1** (Reformulation of Definition 1.1). Let  $\mathcal{M}: \Pi X \rightarrow \mathbb{k}\text{-mod}$  and  $\mathcal{N}: \Pi X^{\text{op}} \rightarrow \mathbb{k}\text{-mod}$  be functors and let  $C_*(\tilde{X}): \Pi X^{\text{op}} \rightarrow \text{Top} \rightarrow \text{Ch}_{\mathbb{k}}$  be the composite of the universal cover functor with the functor  $C_*$ . Define the homology of  $X$  with coefficients in  $\mathcal{M}$  to be

$$H_*(X; \mathcal{M}) = H_*(C_*(\tilde{X}) \otimes_{\Pi X} \mathcal{M})$$

where  $\otimes_{\Pi X}$  is the tensor product of functors (which is given by an evident coequalizer diagram). Similarly, define

$$H^*(X; \mathcal{N}) = H^*\left(\text{Hom}_{\Pi X}(C_*(\tilde{X}), \mathcal{N})\right)$$

where  $\text{Hom}_{\Pi X}$  is the hom of functors (also known as natural transformations; alternatively, given by an evident equalizer diagram).

Note that our distinctions between left and right and between covariant and contravariant are purely semantic above, since we are dealing with groups and groupoids. However, we are about to consider (Bredon) equivariant homology and cohomology. Here the fundamental “groupoid” is only an EI-category (endomorphisms are isomorphisms) and the distinction is essential. There is an equivariant Serre spectral sequence, due to Moerdijk and Svensson [MS93], but it has not yet had significant calculational applications. The essential reason is the lack of a way to compute its  $E^2$ -terms. However, the results of Section 1 generalize nicely to compute Bredon homology and cohomology with local coefficients.

Definition 2.1 generalizes directly to the equivariant case. From now on, let  $X$  be a  $G$ -space, where  $G$  is a discrete group.<sup>1</sup> Following tom Dieck [tD87], we can define the **fundamental EI-category**  $\Pi_G X$  to be the category whose objects are pairs  $(H, x)$ , where  $x \in X^H$ ; a morphism from  $(H, x)$  to  $(K, y)$  consists of a  $G$ -map  $\alpha: G/H \rightarrow G/K$ , determined by  $\alpha(eH) = gK$ , together with a homotopy class  $[\gamma]$  of paths from  $x$  to  $\alpha^*(y) = gy$ . Here  $\alpha^*: X^K \rightarrow X^H$  is the map given by  $\alpha^*(z) = gz$ , which makes sense because  $g^{-1}Hg \subset K$ .

Likewise, we follow tom Dieck in defining the equivariant universal cover  $\tilde{X}$  to be the functor  $\tilde{X}: (\Pi_G X)^{\text{op}} \rightarrow \text{Top}$  which sends  $(H, x)$  to  $\tilde{X}^H(x)$ , the space of equivalence classes of paths in  $X^H$  starting at  $x$ . For a morphism  $(\alpha, [\gamma]): (H, x) \rightarrow (K, y)$ ,  $\tilde{X}(\alpha, [\gamma]): \tilde{X}(K, y) \rightarrow \tilde{X}(H, x)$  takes a class of paths  $[\beta]$  starting at  $y \in X^K$  to the class of the composite  $(\alpha^*\beta)\gamma$ .

We can now define equivariant (co)homology with local coefficients. In fact, Definition 2.1 applies almost verbatim: we need only add  $G$  to the notations. We repeat the definition for emphasis.

**Definition 2.2** (Equivariant generalization of Definition 2.1). Let  $X$  be a  $G$ -space and write  $\Pi = \Pi_G X$ . Let  $\mathcal{M}: \Pi \rightarrow \mathbb{k}\text{-mod}$  and  $\mathcal{N}: \Pi^{\text{op}} \rightarrow \mathbb{k}\text{-mod}$  be functors and let  $C_*^G(\tilde{X}): \Pi^{\text{op}} \rightarrow \text{Top} \rightarrow \text{Ch}_{\mathbb{k}}$  be the composite of the equivariant universal cover functor with the functor  $C_*$ . Define the homology of  $X$  with coefficients in  $\mathcal{M}$  to be

$$H_*^G(X; \mathcal{M}) = H_*(C_*^G(\tilde{X}) \otimes_{\Pi} \mathcal{M})$$

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<sup>1</sup>With a little more detail, we could generalize to topological groups.

and the cohomology of  $X$  with coefficients in  $\mathcal{N}$  by

$$H_G^*(X; \mathcal{N}) = H^* \left( \operatorname{Hom}_{\Pi}(C_*^G(\tilde{X}), \mathcal{N}) \right).$$

Note that we could also take  $\Pi$  to be a skeleton  $\operatorname{skel}(\Pi_G X)$ .

Inserting  $G$  into the notations, the proofs in Whitehead or Hatcher [Whi78, Hat02] apply to show that this definition of Bredon (co)homology with local coefficients is naturally isomorphic to the Bredon (co)homology with local coefficients, as defined in Mukherjee and Pandey [MP01], which they in turn show is naturally isomorphic to the (co)homology with local coefficients, as defined and used by Mørdijk and Svensson in [MS93] to construct the equivariant Serre spectral sequence of a  $G$ -fibration  $f: E \rightarrow B$ .

We quickly review the homological algebra needed for the equivariant generalization of Theorem 1.6. Since  $\mathbb{k}\text{-mod}$  is an Abelian category, the categories  $[\Pi, \mathbb{k}\text{-mod}]$  and  $[\Pi^{\text{op}}, \mathbb{k}\text{-mod}]$  of functors from  $\Pi$  to  $\mathbb{k}\text{-mod}$  are also Abelian, with kernels and cokernels defined levelwise. These categories have enough projectives and injectives, which by the Yoneda lemma are related to the represented functors.

Specifically, let  $R-$  denote the free  $R$ -module functor  $\mathbf{Set} \rightarrow \mathbb{k}\text{-mod}$ . Given an object  $(H, x) \in \Pi$ , let  $P_{H,x}$  be the covariant represented functor  $\Pi \rightarrow \mathbb{k}\text{-mod}$  given on objects by

$$P_{H,x}(K, y) = R\Pi((H, x), (K, y)).$$

By the Yoneda lemma, each  $P_{H,x}$  is projective. Therefore, given a functor  $\mathcal{M}$ , we can construct an epimorphism  $\mathcal{P} \rightarrow \mathcal{M}$  with  $\mathcal{P}$  projective by taking  $\mathcal{P}$  to be a direct sum of representables

$$\mathcal{P} = \bigoplus_{(H,x)} \bigoplus_{\mathcal{M}(H,x)} P_{H,x},$$

one for each element of each  $R$ -module  $\mathcal{M}(H, x)$ . Similarly, there are contravariant represented functors  $P^{H,x}: \Pi^{\text{op}} \rightarrow \mathbb{k}\text{-mod}$  given by

$$P^{H,x}(K, y) = R\Pi((K, y), (H, x)).$$

The same argument shows that these are projective and that  $[\Pi^{\text{op}}, \mathbb{k}\text{-mod}]$  has enough projectives.

The construction of the injective objects is dual but perhaps less familiar. Given an  $R$ -module  $A$  and  $(H, x) \in \Pi$ , we define a functor  $I_{H,x,A}: \Pi \rightarrow \mathbb{k}\text{-mod}$  by

$$I_{H,x,A}(K, y) = \operatorname{Hom}_R(P^{H,x}(K, y), A)$$

Whenever  $A$  is an injective  $R$ -module,  $I_{H,x,A}$  is an injective object in  $[\Pi, \mathbb{k}\text{-mod}]$ . This comes from a more general fact. For any coefficient system  $\mathcal{A}: \Pi \rightarrow \mathbb{k}\text{-mod}$ , there is a tensor-hom adjunction

$$[\Pi, \mathbb{k}\text{-mod}](\mathcal{A}, I_{H,x,A}) \cong \mathbb{k}\text{-mod}(\mathcal{A} \otimes_{\Pi} P^{H,x}, A)$$

where again  $\otimes_{\Pi}$  is the tensor product of functors. The tensor product of any functor with a representable functor  $P^{H,x}$  is given by evaluation at  $(H, x)$ . Putting these two facts together, we have that a natural transformation from  $\mathcal{A}$  to  $I_{H,x,A}$  is given by the same data as a homomorphism of  $R$ -modules from  $\mathcal{A}(H, x)$  to  $A$ . It is then clear that, if  $A$  is an injective  $R$ -module,  $I_{H,x,A}$  must be an injective object of  $[\Pi, \mathbb{k}\text{-mod}]$ , as desired. Given any  $\mathcal{N}: \Pi \rightarrow \mathbb{k}\text{-mod}$ , we can construct an injective coefficient system  $\mathcal{I}$  and a monomorphism  $\mathcal{N} \rightarrow \mathcal{I}$  as follows. Choose

monomorphisms  $\mathcal{N}(H, x) \hookrightarrow A_{H,x}$  for each  $(H, x)$  with  $A_{H,x}$  injective, and define  $\mathcal{J}$  to be the product of injective functors

$$\mathcal{J} = \prod_{(H,x)} I_{H,x,A_{H,x}}.$$

It can be checked that the evident map  $\mathcal{N} \rightarrow \mathcal{J}$  is a monomorphism. Thus  $[\Pi, \mathbb{k}\text{-mod}]$  has enough injectives. The functors  $I^{H,x,A} = \text{Hom}_R(P_{H,x}(-), A)$  show that  $[\Pi^{\text{op}}, \mathbb{k}\text{-mod}]$  has enough injectives as well.

Finally, we define  $\text{Tor}^\Pi(\mathcal{N}, \mathcal{M})$  in the obvious way. It is the homology of the complex of  $R$ -modules that is obtained by taking the tensor product of functors of  $\mathcal{N}$  with a projective resolution of the functor  $\mathcal{M}$ . We define  $\text{Ext}_\Pi(\mathcal{N}_1, \mathcal{N}_2)$  similarly, taking the hom of functors of  $\mathcal{N}_1$  with an injective resolution of  $\mathcal{N}_2$ .<sup>2</sup>

The following equivariant analogue of the nonequivariant statement that  $C_*(\tilde{X})$  is a free  $R[\pi]$ -module should be a standard first observation in equivariant homology theory, but the author has not seen it in the literature. The nonequivariant assertion, while obvious, is the crux of the proof of Theorem 1.6. Let  $P^{H,x} = R\Pi(-, (H, x))$ , as above.

**Lemma 2.3.** *With  $\Pi = \Pi_G X$ , each functor  $C_n^G(\tilde{X}): \Pi^{\text{op}} \rightarrow \mathbb{k}\text{-mod}$  is a direct sum of representable functors  $\bigoplus_{(H_i, x_i)} P^{H_i, x_i}$ .*

Granting this result for the moment, we can prove the equivariant generalization of Theorem 1.6.

**Theorem 2.4** (Equivariant Eilenberg Spectral Sequence). *With  $\Pi = \Pi_G X$ , there are spectral sequences*

$$E_{p,q}^2 = \text{Tor}_{p,q}^\Pi(\mathcal{H}_*(\tilde{X}), \mathcal{M}) \implies H_{p+q}^G(X; \mathcal{M})$$

and

$$E_2^{p,q} = \text{Ext}_\Pi^{p,q}(\mathcal{H}_*(\tilde{X}), \mathcal{N}) \implies H_G^{p+q}(X; \mathcal{N}).$$

Here the functor  $\mathcal{H}_*(\tilde{X}): \Pi^{\text{op}} \rightarrow \mathbb{k}\text{-mod}$  is the homology of the chain complex functor  $C_*^G(\tilde{X})$ ; that is,  $\mathcal{H}_*(\tilde{X})(H, x)$  is the homology of the chain complex  $C_*(\tilde{X})(H, x)$ .

*Proof.* Let  $\varepsilon: \mathcal{P}_* \rightarrow \mathcal{M}$  be a projective resolution of  $\mathcal{M}$ . As in the nonequivariant theorem, form the bicomplex of  $R$ -modules  $C_*(\tilde{X}) \otimes_\Pi \mathcal{P}_*$ . Since the tensor product of a functor with a representable functor is given by evaluation,

$$P^{H,x} \otimes_\Pi \mathcal{M} \cong \mathcal{M}(H, x),$$

tensoring with such projective modules is exact.

In particular, if we filter our bicomplex by degrees of  $C_*(\tilde{X})$ , then  $d^0 = \text{id} \otimes d$ . By Lemma 2.3, each  $C_n^G(\tilde{X})$  is projective, and so we get a spectral sequence with  $E^1$ -term  $C_*(\tilde{X}) \otimes_\Pi \mathcal{M}$ . Thus the resulting  $E^2 = E^\infty$  term is  $H_*(X, \mathcal{M})$ , exactly as in the nonequivariant case.

If we instead filter by degrees of  $\mathcal{P}_*$ , so  $d^0 = d \otimes \text{id}$ , then the  $E^1$  term is  $\mathcal{H}_*(\tilde{X}) \otimes_\Pi \mathcal{P}_*$  and the  $E^2$  term is  $\text{Tor}_{*,*}^\Pi(\mathcal{H}_*(\tilde{X}), \mathcal{M})$ , as desired.

The construction of the second spectral sequence is similar, starting from an injective resolution  $\eta: \mathcal{N} \rightarrow \mathcal{J}^*$ .  $\square$

<sup>2</sup>Alternatively, we could define  $\text{Ext}_\Pi(\mathcal{N}_1, \mathcal{N}_2)$  by taking a projective resolution of  $\mathcal{N}_1$ ; however, it is the definition above which gives rise to the appropriate spectral sequence.



*Proof of Lemma 2.3.* The proof is analogous to that of the nonequivariant result, but more involved. We may identify  $C_n^G(\tilde{X})(H, x)$  with the free  $R$ -module on generators given by the nondegenerate singular  $n$ -simplices  $\sigma: \Delta^n \rightarrow \tilde{X}(H, x)$ . We must show that these free  $R$ -modules piece together appropriately into a free functor. More specifically, by the Yoneda lemma, each  $\sigma: \Delta^n \rightarrow \tilde{X}(K, y)$  determines a natural transformation

$$\iota_\sigma: P^{K, y} \rightarrow C_n^G(\tilde{X})$$

that takes  $\text{id} \in \Pi((K, y), (K, y))$  to  $\sigma$ . We thus obtain a natural transformation

$$\bigoplus_{\{\tau\}} P^{K_\tau, y_\tau} \rightarrow C_n^G(\tilde{X})$$

from any set of nondegenerate  $n$ -simplices  $\{\tau: \Delta^n \rightarrow \tilde{X}(K_\tau, y_\tau)\}$ . We must show that there is a set  $\{\tau\}$  such that the resulting natural transformation is a natural isomorphism, that is, a levelwise isomorphism. This amounts to showing that the following statements hold for our choice of generators  $\tau$  and each object  $(H, x)$ .

- (1) (Injectivity) For any arrows  $(\alpha_1, [\gamma_1])$  and  $(\alpha_2, [\gamma_2])$  in  $\Pi$  with source  $(H, x)$  and any generators  $\tau_1$  and  $\tau_2$ ,  $(\alpha_1, [\gamma_1])^* \tau_1 = (\alpha_2, [\gamma_2])^* \tau_2$  must imply that both  $(\alpha_1, [\gamma_1]) = (\alpha_2, [\gamma_2])$  and  $\tau_1 = \tau_2$ .
- (2) (Surjectivity) For every  $\sigma: \Delta^n \rightarrow \tilde{X}(H, x)$ , there must be a generator  $\tau$  and an arrow  $(\alpha, [\gamma])$  such that  $\sigma = (\alpha, [\gamma])^* \tau$ .

Fixing  $n$ , define the generating set as follows. Regard the initial vertex  $v$  of  $\Delta^n$  as a basepoint. Recall that  $\tilde{X}(K, y)$  is the universal cover of  $X^K$  defined with respect to the basepoint  $y \in X^K$ , so that the equivalence class of the constant path  $c_{K, y}$  at  $y$  is the basepoint of  $\tilde{X}^K$ . In choosing our generating set, we restrict attention to based maps  $\sigma: \Delta^n \rightarrow \tilde{X}(K_\sigma, y_\sigma)$  that are non-degenerate  $n$ -simplices of  $\tilde{X}^{K_\sigma}$ . Such maps  $\sigma$  are in bijective correspondence with based nondegenerate  $n$ -simplices  $\sigma_0: \Delta^n \rightarrow X^{K_\sigma}$ . The correspondence sends  $\sigma$  to its composite with the end-point evaluation map  $p: \tilde{X}(K_\sigma, y_\sigma) \rightarrow X^{K_\sigma}$  and sends  $\sigma_0$  to the map  $\sigma: \Delta^n \rightarrow \tilde{X}(K_\sigma, y_\sigma)$  that sends a point  $a \in \Delta^n$  to the image under  $\sigma_0$  of the straight-line path from  $v$  to  $a$ . Restrict further to those  $\sigma$  that cannot be written as a composite

$$\Delta^n \xrightarrow{\rho} \tilde{X}(K', y') \xrightarrow{(\alpha, \gamma)^*} \tilde{X}(K_\sigma, y_\sigma)$$

for any non-isomorphism  $(\alpha, \gamma): (K', y') \rightarrow (K_\sigma, y_\sigma)$  in  $\Pi$ . Note that, for each such  $\sigma$ , we can obtain another such  $\sigma$  by composing with the isomorphism  $(\xi, \delta)^*$  induced by an isomorphism  $(\xi, \delta)$  in  $\Pi$ . We say that the resulting maps  $\sigma$  are equivalent, and we choose one  $\tau$  in each equivalence class of such based singular  $n$ -simplices  $\sigma$ .

It remains to verify that the natural transformation defined by this set  $\{\tau\}$  is an isomorphism. This is straightforward but somewhat tedious and technical.

For the injectivity, suppose that  $(\alpha_1, [\gamma_1])^* \tau_1 = (\alpha_2, [\gamma_2])^* \tau_2$ , where  $\tau_1, \tau_2$  are in our generating set and

$$\tau_1: \Delta^n \rightarrow \tilde{X}(K_1, y_1), \quad (\alpha_1, [\gamma_1]) \in \Pi((H, x), (K_1, y_1))$$

$$\tau_2: \Delta^n \rightarrow \tilde{X}(K_2, y_2), \quad (\alpha_2, [\gamma_2]) \in \Pi((H, x), (K_2, y_2)).$$

Since  $\tau_i(v) = c_{(K_i, y_i)}$  for  $i = 1, 2$ , we see that  $(\alpha_i, [\gamma_i])^* \tau_i$  must take  $v$  to  $[\gamma_i]$ . Since  $(\alpha_1, [\gamma_1])^* \tau_1 = (\alpha_2, [\gamma_2])^* \tau_2$ , this means that  $[\gamma_1] = [\gamma_2]$ ; call this path class  $[\gamma]$ . In

turn, this implies that  $\alpha_1^* y_1 = \alpha_2^* y_2$ ; call this point  $z \in X^H$ , so that  $[\gamma]$  is a path from  $x$  to  $z$ . Since  $(\alpha_i, [\gamma_i]) = (\alpha_i, [c_z]) \circ (\text{id}, [\gamma])$  and  $(\text{id}, [\gamma])$  is an isomorphism in  $\Pi$ , we must have

$$(\alpha_1, [c_z])^* \tau_1 = (\alpha_2, [c_z])^* \tau_2.$$

In particular, if we compose each side of this equation with  $p$ , we obtain

$$p \circ (\alpha_1, [c_z])^* \tau_1 = p \circ (\alpha_2, [c_z])^* \tau_2$$

as maps  $\Delta^n \rightarrow X^H$ . Since we have commutative diagrams

$$\begin{array}{ccc} \Delta^n & \xrightarrow{\tau_i} & \tilde{X}(K_i, y_i) \xrightarrow{(\alpha, [c_z])^*} \tilde{X}(H, z) \\ & & \downarrow p \quad \quad \downarrow p \\ & & X^{K_i} \xrightarrow{\alpha_i^*} X^H \end{array}$$

for each  $i$ , this implies that we have a commutative square

$$\begin{array}{ccc} \Delta^n & \xrightarrow{p \circ \tau_1} & X^{K_1} \\ p \circ \tau_2 \downarrow & & \downarrow \alpha_1^* \\ X^{K_2} & \xrightarrow{\alpha_2^*} & X^H \end{array}$$

If the maps  $\alpha_i: G/H \rightarrow G/K_i$  are defined by elements  $g_i \in G$ , this implies that the common composite  $\Delta^n \rightarrow X^H$  factors through the fixed-point sets  $X^{g_i K_i g_i^{-1}}$  for each  $i$ , and hence through  $X^L$ , where  $L$  is the smallest subgroup containing  $g_1 K_1 g_1^{-1}$  and  $g_2 K_2 g_2^{-1}$ . Since  $K_i \subset g_i^{-1} L g_i$ , the maps  $\alpha_i: G/H \rightarrow G/K_i$  factor through the maps  $\beta_i: G/K_i \rightarrow G/L$  specified by  $\beta_i(e K_i) = g_i L$  and there result factorizations of the  $\tau_i$  as

$$\Delta^n \longrightarrow \tilde{X}(g_i^{-1} L g_i, y_i) \xrightarrow{(q, [c_z])^*} \tilde{X}(K_i, y_i),$$

where  $q$  denotes either quotient map  $G/K_i \rightarrow G/g_i^{-1} L g_i$ . By our choice of the generators  $\tau$ , this can only happen if  $g_i^{-1} L g_i = K_i$ , giving  $g_1 K_1 g_1^{-1} = g_2 K_2 g_2^{-1}$ . In terms of  $g_1$  and  $g_2$ , we see that our equation  $(\alpha_1, [c_z])^* \tau_1 = (\alpha_2, [c_z])^* \tau_2$  says that  $g_1 \tau_1 = g_2 \tau_2$ , that is,  $\tau_2 = g_2^{-1} g_1 \tau_1$ . Since  $g_2^{-1} g_1$  defines an isomorphism  $G/K_2 \rightarrow G/K_1$ , we again see by our choice of the generators  $\tau$  that  $\tau_1 = \tau_2$  and that  $g_2^{-1} g_1 \in K_1 = K_2$ . This in turn implies that the maps  $\alpha_i: G/H \rightarrow G/K_i$  defined by the  $g_i$  are identical. The conclusion is that  $(\alpha_1, [\gamma_1])^* \tau_1 = (\alpha_2, [\gamma_2])^* \tau_2$  implies  $\tau_1 = \tau_2$  and  $(\alpha_1, [\gamma_1]) = (\alpha_2, [\gamma_2])$ , as desired.

It only remains to show that we have accounted for all elements of  $C_n(\tilde{X})(H, x)$ . For any map  $\sigma: \Delta^n \rightarrow \tilde{X}(H, x)$ ,  $\sigma(v)$  is a homotopy class of paths from  $(H, x)$  to  $(H, x')$  in  $X^H$ . Call this class  $[\gamma]$ . Then  $(\text{id}, [\gamma])$  is an isomorphism with inverse  $(\text{id}, [\gamma^{-1}])$ , and  $\sigma' = (\text{id}, [\gamma^{-1}])^* \sigma$  takes  $v$  to the homotopy class of the constant path at  $(H, x')$ ; it follows that  $\sigma = (\text{id}, [\gamma])^* \sigma'$ . Similarly, if  $\sigma'$  factors through  $\tilde{X}(K, y)$  for some  $K$  properly containing a conjugate of  $H$ , then by definition  $\sigma = (\alpha, [\gamma])^* \tau$  for some  $\tau: \Delta^n \rightarrow \tilde{X}(K, y)$ . We can choose a  $\tau$  that does not itself factor and is in our chosen set of generators.  $\square$

## 3. SOME REMARKS ON THE SERRE SPECTRAL SEQUENCE

We say that a map  $f: E \rightarrow X$  is a  **$G$ -fibration** if  $f^H: E^H \rightarrow X^H$  is a fibration for every subgroup  $H$  of  $G$ . Observe that, if  $x \in X^H$ , then its preimage  $f^{-1}(x) \subset E$  is necessarily an  $H$ -space. As previously mentioned, Moerdijk and Svensson in [MS93] develop an equivariant Serre spectral sequence for  $G$ -fibrations of  $G$ -spaces, which we will now describe.

Given a coefficient system  $M: \mathcal{O}_G^{\text{op}} \rightarrow \mathbb{k}\text{-mod}$  and a subgroup  $H < G$ , there is a restricted coefficient system  $M|_H: \mathcal{O}_H^{\text{op}} \rightarrow \mathbb{k}\text{-mod}$  given by  $M|_H(H/K) := M(G/K)$ . We may thus define a local coefficient system

$$h_G^q(f; M): \Pi_G X^{\text{op}} \rightarrow \mathbf{Ab}$$

to be the functor which acts on objects by

$$(H, x) \mapsto H_G^q(G \times_H f^{-1}(x); M) \cong H_H^q(f^{-1}(x); M|_H).$$

It is defined on morphisms via lifting of paths.

The main result of [MS93] is the following spectral sequence.

**Theorem 3.1** (Moerdijk and Svensson). *For any  $G$ -fibration  $f: E \rightarrow X$  and any coefficient system  $M$ , there is a natural spectral sequence*

$$E_2^{s,t}(M) = H_G^s(X; h_G^t(f; M)) \implies H_G^{s+t}(E; M).$$

Further, this spectral sequence carries a product structure, in the sense that there is a natural pairing of spectral sequences

$$E_r^{s,t}(M) \otimes E_r^{s',t'}(N) \longrightarrow E_r^{s+s',t+t'}(M \otimes N)$$

converging to the standard pairing

$$H_G^*(E; M) \otimes H_G^*(E; N) \xrightarrow{\sim} H_G^*(Y; M \otimes N).$$

On the  $E_2$  page, this pairing agrees with the standard pairing

$$H_G^s(X; h_G^t(f, M)) \otimes H_G^{s'}(X; h_G^{t'}(f, N)) \longrightarrow H_G^{s+s'}(X; h_G^{t+t'}(f, M \otimes N)).^3$$

The tensor product  $M \otimes N$  above is a levelwise tensor product,  $(M \otimes N)(G/H) = M(G/H) \otimes N(G/H)$ , which is distinct from the box product  $\square$  discussed in Section 4.

Although we have been using the integer-graded equivariant cohomology originally defined by Bredon, equivariant cohomology is more naturally graded on  $RO(G)$ . As discussed in [May96, IX], for any coefficient system  $M$  which can be extended to a Mackey functor, the Bredon cohomology theory  $H_G^*(-; M)$  can be extended to an  $RO(G)$ -graded theory. That is, for every virtual representation  $\omega = V - W$ , we have a functor  $H_G^\omega(-; M)$ ; these satisfy appropriate versions of the usual axioms, including suspension:  $\tilde{H}_G^{\omega+V}(\Sigma^V X; M) \cong \tilde{H}_G^\omega(X; M)$  for any honest representation  $V$ . One way of visualizing this extra data is to say that we have one integer-graded theory  $\{H_G^{V+n}\}_{n \in \mathbb{Z}}$  for each representation  $V$  containing no trivial subrepresentations. Each of these theories  $H_G^{V+*}(-; M)$  can be used to define local coefficient systems  $h_G^{V+t}(f, M)$ . Kronholm shows the following in his thesis [Kro09].

<sup>3</sup>As usual in the Serre spectral sequence, the standard pairing on the  $E_2$  page incorporates a sign  $(-1)^{s't}$  relative to the cup product pairing.

**Theorem 3.2** (Kronholm). *For each real representation  $V$ , there is a natural spectral sequence*

$$E_2^{s,t}(M, V) = H_G^s(X; h_G^{V+t}(f, M)) \implies H_G^{V+s+t}(E; M).$$

*Further, for each  $V, V' \in RO(G)$ , there is a pairing*

$$E_r^{s,t}(M, V) \otimes E_r^{s',t'}(N, V') \longrightarrow E_r^{s+s',t+t'}(M \otimes N, V + V')$$

*converging to the standard pairing on  $E_\infty$  and agreeing with the standard pairing on  $E_2$ .*

We have an analogue of Proposition 1.10, as well.

**Proposition 3.3.**  $H_G^0(X; \mathcal{M}) \cong \text{Hom}_\Pi(\underline{R}, \mathcal{M})$ , where  $\underline{R}$  is the constant functor.

*Proof.* As in Proposition 1.10, this comes from identifying  $\mathcal{H}_0(\tilde{X}) \cong \underline{R}$  and from the left exactness of  $\text{Hom}_\Pi$ .  $\square$

**Proposition 3.4.** *Suppose that  $X$  is  $G$ -connected, in the sense that each  $X^H$  is nonempty and connected, and let  $\bullet \in X^G$ . Then  $\text{Hom}_\Pi(\underline{R}, \mathcal{M})$  is isomorphic to a sub- $R$ -module of  $\mathcal{M}(G, \bullet)$ .*

*Proof.* Since  $X$  is  $G$ -connected,  $(G, \bullet)$  is a weakly terminal object in  $\Pi_G X$ , i.e. for every  $(K, y)$  there is a map  $(K, y) \longrightarrow (G, \bullet)$ . It follows that an element of  $\text{Hom}_\Pi(\underline{R}, \mathcal{M})$  is determined by the map of  $R$ -modules  $R \longrightarrow \mathcal{M}(G, \bullet)$ .  $\square$

#### 4. MACKEY FUNCTOR VALUED COHOMOLOGY THEORIES

Preparatory to the computation in Section 7, we will review the theory of Mackey functor valued cohomology theories in this section, and the calculation by Lewis [Lew88] of  $H_G^*(\mathbb{C}P(V))$  in Section 6. Lewis considers cohomology which is not only  $RO(G)$ -graded but Mackey functor valued. He takes coefficients in the Burnside ring Mackey functor  $A = A_G$ . Fix a finite group  $G$ .

**Definition 4.1.** The **Burnside category**  $\mathcal{B}_G$  is the full subcategory of the equivariant stable category on objects  $\Sigma^\infty(\mathbf{b}_+)$ , where  $\mathbf{b}$  is a finite  $G$ -set. Explicitly, the objects of  $\mathcal{B}_G$  are finite  $G$ -sets and the morphisms are the stable  $G$ -maps.

**Definition 4.2.** A **Mackey functor** is a contravariant additive functor from the Burnside category  $\mathcal{B}_G$  to the category  $\mathbb{k}\text{-mod}$ .

For obvious reasons,  $\mathcal{B}_G$  is sometimes called the “stable orbit category” and Mackey functors “stable coefficient systems.” However, it is frequently more convenient to work with the following combinatorial definition. It is shown in [May96] that the two definitions are equivalent for finite  $G$ .

**Definition 4.3.** The category  $\mathcal{B}_G^+$  is the category having:

- objects: the finite  $G$ -sets
- morphisms: equivalence classes of spans

$$\begin{array}{ccc} & \mathbf{u} & \\ \swarrow & & \searrow \\ \mathbf{b} & & \mathbf{c} \end{array}$$

with composition given by pullbacks.

Two spans  $\mathbf{b} \leftarrow \mathbf{u} \rightarrow \mathbf{c}$  and  $\mathbf{b} \leftarrow \mathbf{v} \rightarrow \mathbf{c}$  are equivalent if there is a commutative diagram as follows:

$$\begin{array}{ccc} & \mathbf{u} & \\ \swarrow & \downarrow \cong & \searrow \\ \mathbf{b} & & \mathbf{c} \\ \nwarrow & \downarrow & \nearrow \\ & \mathbf{v} & \end{array}$$

Each hom set  $\mathcal{B}_G^+(\mathbf{b}, \mathbf{c})$  has the structure of an abelian monoid. We may take the sum of  $\mathbf{b} \leftarrow \mathbf{u} \rightarrow \mathbf{c}$  and  $\mathbf{b} \leftarrow \mathbf{v} \rightarrow \mathbf{c}$  to be the span

$$\begin{array}{ccc} & \mathbf{u} \amalg \mathbf{v} & \\ \swarrow & & \searrow \\ \mathbf{b} & & \mathbf{c} \end{array}$$

We may thus apply the Grothendieck construction to the hom sets of  $\mathcal{B}_G^+$ .

**Definition 4.4.** The **Burnside category**  $\mathcal{B}_G$  is the category enriched over  $\mathbf{Ab}$  having:

- objects: the objects of  $\mathcal{B}_G^+$
- morphisms:  $\mathcal{B}_G(\mathbf{b}, \mathbf{c})$  is the Grothendieck group of  $\mathcal{B}_G^+(\mathbf{b}, \mathbf{c})$ .

Since Mackey functors are additive functors, a Mackey functor  $M$  over a commutative ring  $R$  is determined by its values on the orbits  $G/H$ .

**Definition 4.5.** The **Burnside ring Mackey functor**  $A_G$ , abbreviated  $A$  when the group  $G$  is implicit, is the represented functor  $R \otimes \mathcal{B}_G(-, G/G)$ .

The Burnside ring Mackey functor is so named because, when  $R = \mathbb{Z}$ , its value  $A_G(G/H)$  at the orbit  $G/H$  is the underlying Abelian group of the classical Burnside ring  $A(H)$ . In fact, the connection extends to the ring structure as well. The Day tensor product gives a monoidal structure  $\square$  on  $\mathcal{B}_G$ , with unit  $A$ . Explicitly,  $\square$  is a left Kan extension. Given Mackey functors  $M, N: \mathcal{B}_G \rightarrow \mathbf{k}\text{-mod}$ , we can form the external product

$$M \overline{\square} N: \mathcal{B}_G \times \mathcal{B}_G \rightarrow \mathbf{k}\text{-mod}: (\mathbf{b}, \mathbf{c}) \mapsto M(\mathbf{b}) \otimes N(\mathbf{c}),$$

and  $M \square N$  is the left Kan extension of  $M \overline{\square} N$  along the Cartesian product functor  $\times: \mathcal{B}_G \times \mathcal{B}_G \rightarrow \mathcal{B}_G$ :

$$\begin{array}{ccc} \mathcal{B}_G \times \mathcal{B}_G & \xrightarrow{M \overline{\square} N} & \mathbf{k}\text{-mod} \\ \downarrow \times & \nearrow M \square N & \\ \mathcal{B}_G & & \end{array}$$

In other words, natural transformations from  $M \square N$  to another Mackey functor  $P$  are the same as natural transformations from  $M \overline{\square} N$  to  $P \circ \times$ . The upshot of this is that we have a notion of monoids in  $\mathcal{B}_G$ , namely Mackey functors  $T$  together with a “product map”  $T \square T \rightarrow T$  and a unit map  $A \rightarrow T$  satisfying the usual diagrams. These monoids are known as **Mackey functor rings** or **Green functors**. For any Green functor  $T$ , the fact that  $\square$  is a left Kan extension implies that  $T \square T \rightarrow T$  gives each  $T(G/H)$  the structure of an  $R$ -algebra. The Burnside ring Mackey functor  $A$  is, of course, a Green functor, and the ring structure on each  $A(G/H)$  agrees with the ring structure on the classical Burnside ring  $A(H)$ .

The (unstable) orbit category embeds contravariantly and covariantly in  $\mathcal{B}_G$ . Both embeddings are the identity on objects; a map of  $G$ -sets  $G/H \xrightarrow{\alpha} G/K$  is sent to either

$$\begin{array}{ccc} & G/H & \\ \alpha \swarrow & & \searrow \\ G/K & & G/H \end{array} \quad \text{or} \quad \begin{array}{ccc} & G/H & \\ & \searrow \alpha & \\ G/H & & G/K \end{array}$$

as appropriate. Furthermore, every morphism in  $\mathcal{B}_G$  can be written as a composite of such morphisms:

$$\begin{array}{ccc} & \mathbf{u} & \\ \swarrow & & \searrow \\ \mathbf{b} & & \mathbf{c} \end{array} = \begin{array}{ccccc} & \mathbf{u} & & \mathbf{u} & \\ \swarrow & & \searrow & \swarrow & \searrow \\ \mathbf{b} & & \mathbf{u} & & \mathbf{c} \end{array}$$

Hence a Mackey functor  $M$  determines and is determined by a pair of contravariant and covariant coefficient systems which agree on objects (and which satisfy certain compatibility diagrams encoding the composition in  $\mathcal{B}_G$ ). As already mentioned, equivariant homology and cohomology theories whose gradings can be extended to  $RO(G)$  have coefficient systems which extend to Mackey functors.

In particular, we can consider  $RO(G)$ -graded Bredon cohomology with coefficients in the Burnside ring Mackey functor  $A$ ; since  $A$  is the unit for  $\square$ , this is the natural choice of coefficients to consider. In what follows, coefficients in  $A$  will be implicit and omitted from the notation. The  $\mathbb{k}$ -mod valued theory  $H_G^*$  can be extended to a Mackey functor valued theory  $\underline{H}_G^*$  as follows. On orbits,  $\underline{H}_G^*(X)(G/K) := H_G^*(G \times_K X) \cong H_G^*((G/K) \times X) \cong H_K^*(X)$ . On morphisms of type

$$\begin{array}{ccc} & \mathbf{b} & \\ \swarrow & & \searrow \\ \mathbf{b} & & \mathbf{c} \end{array}$$

the required map  $\underline{H}_G^*(X)(\mathbf{c}) \rightarrow \underline{H}_G^*(X)(\mathbf{b})$  is induced on cohomology by the map of spaces  $\mathbf{b} \times X \rightarrow \mathbf{c} \times X$ . On morphisms of type

$$\begin{array}{ccc} & \mathbf{b} & \\ \swarrow & & \searrow \\ \mathbf{c} & & \mathbf{b} \end{array}$$

the map  $\underline{H}_G^*(X)(\mathbf{b}) \rightarrow \underline{H}_G^*(X)(\mathbf{c})$  is induced by an appropriate transfer map  $\Sigma^V \mathbf{c}_+ \rightarrow \Sigma^V \mathbf{b}_+$  and the suspension isomorphism on cohomology. Hence  $\underline{H}_G^*$  ties together information about the equivariant theories  $H_K^*$  for all subgroups  $K < G$ .

## 5. EQUIVARIANT CLASSIFYING SPACES

For the remainder of this paper, with the exception of classical structure groups, groups named with Greek letters will be viewed as structure groups and those named with Latin letters will be viewed as ambient groups of equivariance. Suppose that we are given a structure group  $\Gamma$  and a group of equivariance  $G$ . Then, as discussed in [May96] and many other places, there is a notion of a principal  $(G, \Gamma)$ -bundle, namely a projection to  $\Gamma$ -orbits  $E \rightarrow B = E/\Gamma$  of a  $\Gamma$ -free  $(G \times \Gamma)$ -space  $E$ . Such equivariant bundles are classified by universal principal bundles  $E_G \Gamma \rightarrow B_G \Gamma$ , where  $E_G \Gamma$  is a space whose fixed point sets  $(E_G \Gamma)^\Lambda$  are empty when  $\Lambda \subset G \times \Gamma$  intersects  $\Gamma = \{e\} \times \Gamma$  nontrivially and contractible when  $\Lambda \cap \Gamma = \{e\}$ .

As should be expected, for a fixed group  $G$ , the equivariant classifying space construction can be made functorial; that is, there are functors

$$\begin{aligned} E_G &: \mathbf{Grp} \longrightarrow G\text{-Top} \\ B_G &: \mathbf{Grp} \longrightarrow G\text{-Top}. \end{aligned}$$

It will be helpful to pick particular functors  $E_G$  and  $B_G$ , using the categorical two-sided bar construction. Given any groups  $G$  and  $\Gamma$ , we may take

$$\begin{aligned} E_G \Gamma &:= B(T_\Gamma, \mathcal{O}_{G \times \Gamma}, O_{G \times \Gamma}) \\ B_G \Gamma &:= (E_G \Gamma) / \Gamma \end{aligned}$$

where  $\mathcal{O}_{G \times \Gamma}$  is the orbit category,  $O_{G \times \Gamma}: \mathcal{O}_{G \times \Gamma} \longrightarrow \text{Top}$  is given by viewing an orbit  $(G \times \Gamma) / \Lambda$  as a topological space, and  $T_\Gamma: \mathcal{O}_{G \times \Gamma}^{\text{op}} \longrightarrow \text{Top}$  is the functor which takes

$$(G \times \Gamma) / \Lambda \longmapsto \begin{cases} \bullet & \text{if } \Lambda \cap \Gamma = \{e\} \\ \emptyset & \text{otherwise} \end{cases}$$

Since the functor  $O_{G \times \Gamma}$  lands in  $(G \times \Gamma)\text{-Top}$ ,  $E_G \Gamma$  is a  $(G \times \Gamma)$ -space, and it is easy to check that it has the correct fixed points.

The bar construction  $B(-, -, -)$  is a functor from the category of triples  $(T, \mathcal{C}, S)$  to  $\text{Top}$ . Here  $\mathcal{C}$  is a category and  $S, T$  are respectively a covariant and a contravariant functor  $\mathcal{C} \longrightarrow \text{Top}$ . A morphism  $(T_1, \mathcal{C}_1, S_1) \longrightarrow (T_2, \mathcal{C}_2, S_2)$  in the category of triples consists of a functor  $F: \mathcal{C}_1 \longrightarrow \mathcal{C}_2$  together with natural transformations  $S_1 \longrightarrow S_2 \circ F$  and  $T_1 \longrightarrow T_2 \circ F^{\text{op}}$ . It follows that, for fixed  $G$ , we can make  $E_G(-)$  into a functor  $\mathbf{Grp} \longrightarrow \text{Top}$  as follows. Given a homomorphism  $\varphi: \Gamma_1 \longrightarrow \Gamma_2$ , we apply  $B(-, -, -)$  to the morphism of triples given by the functor

$$F: \mathcal{O}_{G \times \Gamma_1} \longrightarrow \mathcal{O}_{G \times \Gamma_2}: (G \times \Gamma_1) / \Lambda \longmapsto (G \times \Gamma_2) / ((\text{id} \times \varphi)(\Lambda)),$$

with the obvious natural transformations  $O_{G \times \Gamma_1} \longrightarrow O_{G \times \Gamma_2} \circ F$  and  $T_{\Gamma_1} \longrightarrow T_{\Gamma_2} \circ F^{\text{op}}$  (for the latter, note that if  $\Lambda \cap \Gamma_1 = \{e\}$ , then also  $(\text{id} \times \varphi)(\Lambda) \cap \Gamma_2 = \{e\}$ ). Since the morphism  $E_G \Gamma_1 \longrightarrow E_G \Gamma_2$  induced by  $\Gamma_1 \longrightarrow \Gamma_2$  is  $\Gamma_1$ -equivariant, there is an induced map  $B_G \Gamma_1 \longrightarrow B_G \Gamma_2$ , making  $B_G(-)$  a functor.

The following result will be useful later.

**Proposition 5.1.** *Fix a group  $G$ . Corresponding to any short exact sequence of structure groups*

$$1 \longrightarrow \Gamma \xrightarrow{\varphi} \Upsilon \xrightarrow{\psi} \Sigma \longrightarrow 1$$

*there is a pullback square in the category of  $G$ -spaces*

$$\begin{array}{ccc} B_G \Gamma & \longrightarrow & E_G \Sigma \\ \downarrow \lrcorner & & \downarrow / \Sigma \\ B_G \Upsilon & \xrightarrow{B_G \psi} & B_G \Sigma \end{array}$$

*Proof.* By functoriality of  $E_G(-)$  and the definition of  $B_G(-)$ , the map  $\psi$  induces a commutative diagram

$$\begin{array}{ccc} E_G \Upsilon & \xrightarrow{E_G \psi} & E_G \Sigma \\ \downarrow / \Upsilon & & \downarrow / \Upsilon = / \Sigma \\ B_G \Upsilon & \xrightarrow{B_G \psi} & B_G \Sigma \end{array}$$

Further, the projection  $E_G \Upsilon \longrightarrow B_G \Upsilon = (E_G \Upsilon)/\Upsilon$  factors as

$$E_G \Upsilon \longrightarrow (E_G \Upsilon)/\Gamma \xrightarrow{/\Sigma} (E_G \Upsilon)/\Upsilon$$

and since the action of  $\Gamma$  on  $E_G \Sigma$  is trivial,  $E_G \Upsilon \longrightarrow E_G \Sigma$  also factors through  $(E_G \Upsilon)/\Gamma \cong B_G \Gamma$ . We thus get the commutative square described in the proposition. Viewing  $E_G \Sigma$  as a  $G$ -space via the inclusion  $G \hookrightarrow G \times \Sigma$ , this is a diagram in the category of  $G$ -spaces. The square induces a homeomorphism on the fibers of the vertical maps, and hence is a pullback in the category of  $G$ -spaces, as desired.  $\square$

## 6. THE COHOMOLOGY OF COMPLEX PROJECTIVE SPACES

Now specialize to the case  $G = C_p$ , the cyclic group of order  $p$ , for some prime  $p > 2$ . In [Lew88], Lewis shows that certain  $C_p$ -spaces, including the complex projective spaces  $\mathbb{CP}(V)$  on an arbitrary  $C_p$ -representation  $V$ , have cohomology which is additively free as a module over the cohomology of a point  $\underline{H}_{C_p}^* := \underline{H}_{C_p}^*(\bullet)$ .<sup>4</sup> The cohomology of a point is a Green functor, with monoid structure induced by the diagonal map  $\bullet \longrightarrow \bullet \times \bullet$ , so it makes sense to talk about modules over it. As usual, such a module is called free if it is of the form  $\underline{H}_{C_p}^* \square \mathcal{B}_{C_p}(-, \mathbf{b})$ , i.e. a box product of  $\underline{H}_{C_p}^*$  and a representable Mackey functor. Further, for the complex projective spaces  $\mathbb{CP}(V)$ , Lewis describes the product structure

$$\underline{H}_{C_p}^*(\mathbb{CP}(V)) \square \underline{H}_{C_p}^*(\mathbb{CP}(V)) \longrightarrow \underline{H}_{C_p}^*(\mathbb{CP}(V))$$

which we will now summarize in one case of interest.<sup>5</sup>

Consider the complete universe  $\mathcal{U}_{\mathbb{C}}$ , a direct sum of countably infinitely many copies of each irreducible complex representation of  $C_p$ .  $\mathbb{CP}(\mathcal{U}_{\mathbb{C}})$  is a space of interest because, as will be discussed in Section 7, it is a model for the equivariant classifying space  $B_{C_p} SO(2)$  and hence is related to equivariant characteristic classes. We can describe  $\mathcal{U}_{\mathbb{C}}$  explicitly: the irreducibles all have complex dimension one, and a generator  $g$  of  $C_p$  acts by rotation through  $2\pi j/p$  radians in the complex plane, for some integer  $j$ . Hence there are  $p$  irreducible representations  $\phi_0, \phi_1, \dots, \phi_{p-1}$ , with  $\phi_0$  trivial and  $\phi_1$  through  $\phi_{p-1}$  nontrivial. For convenience we may assume that  $\phi_j = \phi_1^j$  (i.e.  $\phi_1^{\otimes j}$ ) for each  $j$ . Since  $\phi_1^p = \phi_0$ , it then also makes sense to write  $\phi_j = \phi_1^j$  for  $j > p$ . We have  $\phi_j = \phi_{j+p} = \phi_{j+2p} = \dots$ , and we can write

$$\mathcal{U} = \phi_0 + \phi_1 + \dots + \phi_{p-1} + \phi_p + \dots.$$

The dimensions of the generators of  $\underline{H}_{C_p}^*(\mathbb{CP}(\mathcal{U}_{\mathbb{C}}))$  are related to the underlying real representations of the  $\phi_j$ ; it is worth giving these dimensions their own names.

**Definition 6.1.** Let  $\omega_0$  be the zero-dimensional representation 0. For each positive integer  $j$ , define  $\omega_j$  to be the underlying real representation of

$$\phi_j^{-1}(\phi_0 + \phi_1 + \dots + \phi_{j-1})$$

Observe that  $\omega_p$  is the underlying real representation of the regular complex representation  $\lambda$ . More generally, if  $j = rp + j_0$  and  $0 \leq j_0 < p$ , then  $\omega_j$  is the underlying real representation of  $r\lambda \oplus \phi_{p-j_0} \oplus \dots \oplus \phi_{p-1}$ .

<sup>4</sup>[Lew88] also covers the case  $p = 2$ ; the details are actually easier but slightly different.

<sup>5</sup>This paper is part of the author's Ph.D. thesis, which discusses Lewis's calculation in more detail and correct several errors.



**Theorem 6.2** (Lewis). *As an  $RO(G)$ -graded module over  $\underline{H}_{C_p}^*$ ,*

$$\underline{H}_{C_p}^*(\mathbb{CP}(\mathcal{U}_{\mathbb{C}})) \cong \bigoplus_{j \geq 0} \Sigma^{\omega_j} \left( \underline{H}_{C_p}^* \square \mathcal{B}_{C_p}(-, C_p/C_p) \right) \cong \bigoplus_{j \geq 0} \Sigma^{\omega_j} \underline{H}_{C_p}^*$$

where the  $\omega_j$  are as given in Definition 6.1.

Label the generators in dimensions  $\omega_1$  through  $\omega_{p-1}$  as  $D_1$  through  $D_{p-1}$ , and let  $C$  be the generator in dimension  $\omega_p$ . It is shown in [Lew88] that these elements generate  $\underline{H}_{C_p}^*(\mathbb{CP}(\mathcal{U}_{\mathbb{C}}))$  as an algebra over  $\underline{H}_{C_p}^*$ .

Explicitly,  $C$  is a map of  $\underline{H}_{C_p}^*$ -modules  $\Sigma^{\omega_p}(\underline{H}_{C_p}^* \square \mathcal{B}_{C_p}(-, C_p/C_p)) \rightarrow \underline{H}_{C_p}^*(\mathbb{CP}(\mathcal{U}_{\mathbb{C}}))$ , which by Yoneda we may identify with an element of

$$\underline{H}_{C_p}^{\omega_p}(\mathbb{CP}(\mathcal{U}_{\mathbb{C}}))(C_p/C_p) \cong H_{C_p}^{\omega_p}(\mathbb{CP}(\mathcal{U}_{\mathbb{C}})),$$

and similarly for the  $D_j$ . To describe the product structure on the graded Mackey functor  $\underline{H}_{C_p}^*(\mathbb{CP}(\mathcal{U}_{\mathbb{C}}))$ , it suffices to describe the levelwise products

$$\underline{H}_{C_p}^*(\mathbb{CP}(\mathcal{U}_{\mathbb{C}}))(C_p/H) \otimes \underline{H}_{C_p}^*(\mathbb{CP}(\mathcal{U}_{\mathbb{C}}))(C_p/H) \rightarrow \underline{H}_{C_p}^*(\mathbb{CP}(\mathcal{U}_{\mathbb{C}}))(C_p/H)$$

for each  $H < C_p$ . However,  $C_p$  has only two subgroups. If  $H = \{e\}$ , then  $\underline{H}_{C_p}^*(\mathbb{CP}(\mathcal{U}_{\mathbb{C}}))(C_p/e) \cong H^{|\ast|}(\mathbb{CP}^{\infty})$  with the expected product structure. If  $H = C_p$ , we get  $RO(C_p)$ -graded Bredon cohomology, where  $C$  and the  $D_j$  live.

It remains to describe the products of the  $D_j$  and  $C$ . For notational convenience, let  $D_0 = 1$ .

**Theorem 6.3** (Lewis). *As a commutative algebra over  $\underline{H}_{C_p}^*$ ,  $\underline{H}_{C_p}^*(\mathbb{CP}(\mathcal{U}_{\mathbb{C}}))$  is generated by elements  $C$  in dimension  $\omega_p$  and  $D_j$  in dimension  $\omega_j$  for each  $1 \leq j \leq p-1$ .  $C$  generates a polynomial subalgebra of  $\underline{H}_{C_p}^*(\mathbb{CP}(\mathcal{U}_{\mathbb{C}}))$ , and a complete set of additive generators of  $\underline{H}_{C_p}^*(\mathbb{CP}(\mathcal{U}_{\mathbb{C}}))$  is given by the elements  $\{D_j C^n\}_{0 \leq j \leq p-1, n \geq 0}$ . If  $j \leq k \leq p-1$ , then each product  $D_j D_k = D_k D_j$  is given by a linear combination over  $\underline{H}_{C_p}^*$  of the elements  $D_j$  through  $D_{j+k}$ , if  $j+k \leq p-1$ , and of  $D_j$  through  $D_{p-1}$  and  $D_0 C = C$  through  $D_{j+k-p} C$  if  $j+k \geq p$ .*

The list of relations takes several pages to write out, and so will not be reproduced here; see [Lew88] for details. Some things to note are that  $D_j \neq D_1^j$  (the dimensions are not correct), and that  $D_1^p$  is a nontrivial linear combination (over  $\underline{H}_{C_p}^*$ ) of the  $D_j$  and  $C$ .

The elements  $C$  and the  $D_j$  are generators for  $\underline{H}_{C_p}^* \mathbb{CP}(\mathcal{U}_{\mathbb{C}})$  as a graded Green functor. However, since these generators all live at the  $C_p/C_p$  level, there are no complications; they give a set of algebra generators for the  $RO(G)$ -graded,  $\mathbb{k}$ -mod valued theory  $\underline{H}_{C_p}^* \mathbb{CP}(\mathcal{U}_{\mathbb{C}})$  as well.

One final note: Lewis uses  $R = \mathbb{Z}$  and the Burnside ring Mackey functor  $A$  for all of his calculations. However, if we instead used  $R = \mathbb{F}_q$  and  $A \cong A \otimes \mathbb{F}_q$  (with the tensor product taken levelwise) for any prime  $q \neq p$ , all arguments go through verbatim, and the algebra structure looks the same. In the case  $q = p$ , however, the crucial [Lew88, Corollary 2.7] fails, and so the answer looks very different.

## 7. EXAMPLE: THE COHOMOLOGY OF $B_{C_p}O(2)$

As before, let  $C_p$  be the cyclic group of order  $p$ . In this section, we will calculate the  $RO(C_p)$ -graded Bredon cohomology

$$H_{C_p}^*(B_{C_p}O(2); A \otimes \mathbb{F}_q)$$

of the equivariant classifying space  $B_{C_p}O(2)$  for odd primes  $p \neq q$ . By way of motivation, we know from [Lew88] what  $H_{C_p}^* B_{C_p} S^1$  is, and historically  $O(2)$  is often the first test case to try after  $S^1$ . Let  $D_1$  through  $D_p$  and  $C$  be the elements described in Theorem 6.3.

**Theorem 7.1.** *Let  $p, q$  be distinct odd primes. Then, as an algebra over  $H_{C_p}^*$ ,  $H_{C_p}^*(B_{C_p}O(2); A \otimes \mathbb{F}_q)$  is isomorphic to the subalgebra of  $H_{C_p}^*(\mathbb{C}P(\mathcal{U}_{\mathbb{C}}); A \otimes \mathbb{F}_q)$  generated by the elements  $D_2, D_4, \dots, D_{p-1}, D_1 C, \dots, D_{p-2} C, C^2$ .*

By analogy with Example 1.11, we will approach Theorem 7.1 via the short exact sequence of structure groups

$$1 \longrightarrow SO(2) \longrightarrow O(2) \longrightarrow \mathbb{Z}/2 \longrightarrow 1$$

and the induced fibration  $f: B_{C_p}O(2) \longrightarrow B_{C_p}\mathbb{Z}/2$ , where again  $\mathbb{Z}/2$  is the cyclic group of order 2. We will first identify the  $C_p$ -action on the fibers of  $f$ , then explicitly describe the coefficient systems  $h^{V+t}(f, A)$  and  $\mathcal{H}_*(\widetilde{B_{C_p}\mathbb{Z}/2})$ , and finally prove the theorem.

**7.1. Identifying the fibers of  $f: B_{C_p}O(2) \longrightarrow B_{C_p}\mathbb{Z}/2$ .** We will begin by identifying models for the equivariant classifying spaces under consideration. Recall that nonequivariantly, the universal bundle  $E\mathbb{Z}/2 \longrightarrow B\mathbb{Z}/2$  has as a model  $S^\infty \longrightarrow \mathbb{R}P^\infty$ , where  $S^\infty = S(\mathbb{R}^\infty)$  is the unit sphere in  $\mathbb{R}^\infty$  and  $\mathbb{R}P^\infty = \mathbb{R}P(\mathbb{R}^\infty)$  is the infinite-dimensional real projective space. In general, for any group  $G$ , let  $\mathcal{U}_{\mathbb{R}}$  be a direct sum containing countably infinitely many copies of each real representation of  $G$ . Then  $S(\mathcal{U}_{\mathbb{R}}) \longrightarrow \mathbb{R}P(\mathcal{U}_{\mathbb{R}})$  is a model for  $E_G\mathbb{Z}/2 \longrightarrow B_G\mathbb{Z}/2$ . The  $G \times \mathbb{Z}/2$  action on  $S(\mathcal{U}_{\mathbb{R}})$  comes from the  $G$  action on  $\mathcal{U}_{\mathbb{R}}$  and the  $\mathbb{Z}/2$  action by multiplication by  $-1$ . When  $G = C_p$  is cyclic of prime order, however, we can choose a simpler model.

**Lemma 7.2.** *If  $p$  is an odd prime, then  $E_{C_p}\mathbb{Z}/2 \longrightarrow B_{C_p}\mathbb{Z}/2$  has as a model  $S^\infty \longrightarrow \mathbb{R}P^\infty$  with the trivial  $C_p$ -action on both spaces.*

*Proof.* To verify this claim, it suffices to check the fixed-point sets of  $S^\infty$ . Since  $\mathbb{Z}/2$  acts freely on  $S^\infty$ , the fixed points  $(S^\infty)^\Lambda$  are certainly empty when  $\Lambda \cap \mathbb{Z}/2$  is nontrivial. So we need only check that the fixed point set is contractible whenever  $\Lambda \cap \mathbb{Z}/2$  is trivial.

Note that the subgroups  $\Lambda \subset C_p \times \mathbb{Z}/2$  which intersect  $\mathbb{Z}/2$  trivially are the “twisted diagonal subgroups”  $\Lambda = \Delta_{\rho, H} = \{(h, \rho(h)) \mid h \in H\}$ , for  $H$  a subgroup of  $C_p$  and  $\rho: H \longrightarrow \mathbb{Z}/2$  a homomorphism. However, since  $p$  is an odd prime, the only homomorphism  $H \longrightarrow \mathbb{Z}/2$  is the trivial homomorphism, and so  $\Delta_{\rho, H} = H \times \{e\}$ . This acts trivially on  $S^\infty$ , so  $(S^\infty)^\Lambda = S^\infty \simeq \bullet$ , as desired.  $\square$

Similarly,  $SO(2)$  is the circle  $\mathbb{T}$ . Letting  $\mathcal{U}_{\mathbb{C}}$  again be the direct sum of countably infinitely many copies of each irreducible complex representation of  $C_p$ , an analysis of the fixed-point sets of  $S(\mathcal{U}_{\mathbb{C}})$  gives the following well-known result.

**Lemma 7.3.** *For any prime  $p$ ,  $E_{C_p}SO(2) \longrightarrow B_{C_p}SO(2)$  has as a model  $S(\mathcal{U}_{\mathbb{C}}) \longrightarrow \mathbb{C}P(\mathcal{U}_{\mathbb{C}})$ . The  $C_p \times SO(2)$  action on  $S(\mathcal{U}_{\mathbb{C}})$  comes from the  $C_p$  action on  $\mathcal{U}_{\mathbb{C}}$  and the usual circle action on the complex plane.*  $\square$

We are now in a position to use Proposition 5.1 for the short exact sequence

$$1 \longrightarrow SO(2) \longrightarrow O(2) \xrightarrow{\det} \mathbb{Z}/2 \longrightarrow 1.$$

We have a pullback square in the category of  $C_p$ -spaces

$$\begin{array}{ccc} B_{C_p} SO(2) & \longrightarrow & E_{C_p} \mathbb{Z}/2 \\ \downarrow \lrcorner & & \downarrow \\ B_{C_p} O(2) & \longrightarrow & B_{C_p} \mathbb{Z}/2 \end{array}$$

By Lemma 7.2, the  $C_p$ -actions on  $E_{C_p} \mathbb{Z}/2$  and  $B_{C_p} \mathbb{Z}/2$  are trivial. It follows that  $E_{C_p} \mathbb{Z}/2$  is  $C_p$ -contractible, and so we have proved the following about our map  $f: B_{C_p} O(2) \longrightarrow B_{C_p} \mathbb{Z}/2$ .

**Lemma 7.4.** *For each point  $x \in B_{C_p} \mathbb{Z}/2$ , the fiber  $f^{-1}(x)$  is  $C_p$ -homotopy equivalent to  $B_{C_p} SO(2) = \mathbb{C}P(\mathcal{U}_{\mathbb{C}})$ .  $\square$*

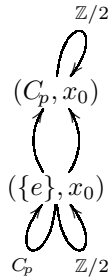
**7.2. The local coefficient system  $h_{C_p}^{V+t}(f, A)$ .** Recall from Theorem 3.2 that the equivariant Serre spectral sequence for a  $C_p$ -fibration  $f: E \longrightarrow X$  has

$$E_2^{s,t}(M, V) = H_{C_p}^s(X; h_{C_p}^{V+t}(f, M)) \implies H_{C_p}^{V+s+t}(E; M).$$

Choose the coefficient ring  $R = \mathbb{F}_q$ , the finite field with  $q$  elements, for an odd prime  $q \neq p$ . As in Theorem 7.1, let  $M = A$ ; note that  $A$  here is a represented functor to  $\mathbb{F}_q\text{-mod}$ , but we choose not to explicitly make the  $q$  part of the notation. We must first analyze the local coefficient systems

$$h_{C_p}^{V+t}(f, A): (H, x) \longmapsto H_{C_p}^{V+t}(C_p \times_H f^{-1}(x)).$$

We may start by taking a skeleton  $\Pi$  of the category  $\Pi_{C_p} B_{C_p} \mathbb{Z}/2$ . Again using Lemma 7.2, we see that each fixed-point set  $(B_{C_p} \mathbb{Z}/2)^H$  is nonempty and connected with fundamental group  $\mathbb{Z}/2$ , so  $\Pi$  has two objects  $(C_p, x_0)$  and  $(\{e\}, x_0)$ . Recalling that a map  $(H, x) \longrightarrow (K, y)$  consists of a  $C_p$ -map  $\alpha: C_p/H \longrightarrow C_p/K$  and a homotopy class of paths  $[\gamma]$  from  $x$  to  $\alpha^*y$ , we see that there are two endomorphisms of  $(C_p, x_0)$ . One of these is the identity, and the other,  $\kappa$ , squares to the identity. Similarly, there are two morphisms  $(\{e\}, x_0) \longrightarrow (C_p, x_0)$ , and composition with  $\kappa$  exchanges them. Finally, the endomorphisms of  $(\{e\}, x_0)$  are in bijection with  $C_p \times \mathbb{Z}/2$ ; when precomposing with the two morphisms  $(\{e\}, x_0) \longrightarrow (C_p, x_0)$ , only the  $\mathbb{Z}/2$  factor has an effect. We may visualize  $\Pi$  as follows:



We can then explicitly describe the coefficient system  $h_{C_p}^{V+t}(f, A)$ . Let  $\text{proj}_{C_p}$  be the projection  $C_p/\{e\} \longrightarrow C_p/C_p$ .

**Proposition 7.5.** *The functor  $h_{C_p}^{V+t}(f, A): \Pi \longrightarrow \mathbb{k}\text{-mod}$  takes*

$$\begin{aligned} (C_p, x_0) &\longmapsto H_{C_p}^{V+t}(\mathbb{CP}(\mathcal{U}_{\mathbb{C}}); A) = \underline{H}_{C_p}^{V+t}(\mathbb{CP}(\mathcal{U}_{\mathbb{C}}))(C_p/\{e\}) \\ (\{e\}, x_0) &\longmapsto H^{|V|+t}(\mathbb{CP}(\mathcal{U}_{\mathbb{C}}); A(C_p/e)) = \underline{H}_{C_p}^{V+t}(\mathbb{CP}(\mathcal{U}_{\mathbb{C}}))(C_p/C_p) \end{aligned}$$

*The functor is determined on morphisms by the following:*

- (1) *the image of  $(\text{proj}_{C_p}, [c_{(\{e\}, x_0)}])$  is the image of  $\text{proj}_{C_p}$  in the underlying contravariant coefficient system of the Mackey functor  $\underline{H}_{C_p}^{V+t}(\mathbb{CP}(\mathcal{U}_{\mathbb{C}}))$ ;*
- (2) *for each map  $g: C_p/\{e\} \longrightarrow C_p/C_p$ , the image of  $(g, [c_{(\{e\}, x_0)}])$  is the image of  $g$  in the underlying contravariant coefficient system of  $\underline{H}_{C_p}^{V+t}(\mathbb{CP}(\mathcal{U}_{\mathbb{C}}))$ ;*
- (3) *Let  $D_1$  through  $D_{p-1}$  and  $C$  be the algebra generators described in Theorem 6.3. The nontrivial automorphism of  $(C_p, x_0)$  acts by the identity on the  $D_{2k}$  and by multiplication by  $-1$  on the  $D_{2k-1}$  and  $C$ .*

This may be visualized by the diagram below.

$$\begin{array}{c} \mathbb{Z}/2 \\ \curvearrowright \\ H_{C_p}^{V+t}(\mathbb{CP}(\mathcal{U}_{\mathbb{C}}); A) \\ \downarrow \\ H^{|V|+t}(\mathbb{CP}(\mathcal{U}_{\mathbb{C}}); A(C_p/e)) \\ \downarrow \\ C_p \quad \mathbb{Z}/2 \end{array}$$

Note that the second downward arrow is given by composing the maps of (1) and (3).

*Proof.* Lemma 7.4 identifies the value of  $h_{C_p}^{V+t}(f, A)$  on objects.

For item (1), the downward arrow induced by the morphism  $(\text{proj}_{C_p}, [c_{(\{e\}, x_0)}])$  is simply the map on cohomology induced by the space-level map

$$C_p \times f^{-1}(x) \longrightarrow f^{-1}(x).$$

This is the same as the map  $\underline{H}_{C_p}^{V+t}(\mathbb{CP}(\mathcal{U}_{\mathbb{C}}))(C_p/C_p) \longrightarrow \underline{H}_{C_p}^{V+t}(\mathbb{CP}(\mathcal{U}_{\mathbb{C}}))(C_p/\{e\})$  induced by the span

$$\begin{array}{ccc} & C_p/\{e\} & \\ \swarrow & & \searrow \\ C_p/\{e\} & \xlongequal{\quad} & C_p/C_p \end{array}$$

A similar argument identifies the map in item (2).

It remains to identify the  $\mathbb{Z}/2$  action. Recall that any  $B_G\Pi$  is of the nonequivariant homotopy type of the classifying space  $B\Pi$ . In particular, our fibration  $B_{C_p}O(2) \longrightarrow B_{C_p}\mathbb{Z}/2$  corresponds to the nonequivariant fiber sequence

$$BSO(2) \longrightarrow BO(2) \xrightarrow{B\det} B\mathbb{Z}/2$$

We know that  $H^*(BSO(2); \mathbb{F}_q) \cong \mathbb{F}_q[x]$ , a polynomial algebra on a generator  $x$  in degree 2, and that  $\pi_1 B\mathbb{Z}/2$  acts by  $-1$  on  $x$ . This determines the  $\mathbb{Z}/2$  action at the  $C_p/\{e\}$  level in the Mackey functor  $\underline{H}_{C_p}^{V+t}$ , and thus at the  $C_p/\{e\}$  level in  $h_{C_p}^{V+t}(f, A)$ . The algebra generators of  $\underline{H}_{C_p}^*(\mathbb{CP}(\mathcal{U}_{\mathbb{C}}))$  at the  $C_p/C_p$  level are  $D_1$

through  $D_{p-1}$  and  $C$ , where  $D_j$  restricts to  $x^j$  at the  $C_p/\{e\}$  level and  $C$  restricts to  $x^p$ . It follows that the action of  $\mathbb{Z}/2$  at the  $C_p/C_p$  level must be by  $-1$  on the elements  $D_{2k-1}$  and  $C$ , and by the identity on the  $D_{2k}$ .  $\square$

**7.3. The local coefficient system  $\mathcal{H}_*(\widetilde{B_{C_p}\mathbb{Z}/2})$ .** Recall that  $\mathcal{H}_*(\tilde{X})$  is the coefficient system  $\Pi X^{\text{op}} \rightarrow \mathbf{k}\text{-mod}$  which takes  $(H, x) \mapsto H_*(\tilde{X}^H(x))$ . In our case, since  $B_{C_p}\mathbb{Z}/2 \cong \mathbb{R}P^\infty$  with trivial  $C_p$ -action, it follows that  $\mathcal{H}_*(\widetilde{B_{C_p}\mathbb{Z}/2})$  is the constant functor at  $H_*(S^\infty) \cong H_*(\bullet)$ . We will continue to take our coefficient ring  $R = \mathbb{F}_q$  for  $q \neq p$ , so  $H_*(\bullet) \cong \mathbb{F}_q$  concentrated in dimension 0.

One reason that  $R = \mathbb{F}_q$  is such a convenient choice in this example is that the constant functor at  $H_*(\bullet)$  is projective.

**Proposition 7.6.** *The constant functor  $\underline{\mathbb{F}}_q: \Pi^{\text{op}} \rightarrow \mathbb{F}_q\text{-mod}$  is a direct summand of a representable functor and hence a projective object in the category  $[\Pi^{\text{op}}, \mathbb{F}_q\text{-mod}]$ .*

*Proof.* Consider the represented functor  $\mathbb{F}_q\Pi(-, (\bullet, x_0))$ . We see by inspection that  $\mathbb{F}_q\Pi((K, x_0), (\bullet, x_0)) \cong \mathbb{F}_q[\mathbb{Z}/2]$  for both possible values of  $K$ . For an appropriate choice of basis, we can display this as

$$\begin{array}{c}
 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
 \downarrow \\
 \mathbb{F}_q \oplus \mathbb{F}_q \\
 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \downarrow \quad \downarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
 \mathbb{F}_q \oplus \mathbb{F}_q \\
 \downarrow \quad \downarrow \\
 \text{triv} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
 \end{array}$$

That is, the nontrivial element of  $\mathbb{Z}/2$  acts by interchanging the basis elements, on both the top and the bottom. The action of  $C_p$  on the bottom is trivial, and the downward maps behave as shown. If we take the new basis given by the change-of-coordinates matrix  $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$  (using the fact that  $q \neq 2$ ), we see that  $\mathbb{F}_q\Pi(-, (\bullet, x_0))$  breaks up as the direct sum of two functors, one of which is our constant functor  $\underline{\mathbb{F}}_q$ .

$$\mathbb{F}_q\Pi(-, (\bullet, x_0)) \cong \begin{array}{c} \text{id} \\ \downarrow \\ \mathbb{F}_q \\ \downarrow \quad \downarrow \\ \mathbb{F}_q \\ \downarrow \quad \downarrow \\ \text{triv} \quad \text{id} \end{array} \oplus \begin{array}{c} -1 \\ \downarrow \\ \mathbb{F}_q \\ \downarrow \quad \downarrow \\ \mathbb{F}_q \\ \downarrow \quad \downarrow \\ \text{triv} \quad -1 \end{array}$$

$\square$

**7.4. The calculation of  $H_{C_p}^*(B_{C_p}O(2); A \otimes \mathbb{F}_q)$ .** We are now prepared to prove Theorem 7.1. Fix odd primes  $p \neq q$ . We will continue to make heavy use of the identification of  $B_{C_p}\mathbb{Z}/2$  in Lemma 7.2.

*Proof of Theorem 7.1.* We will use the equivariant Eilenberg spectral sequence to identify the  $E_2$  page of the Serre spectral sequence for  $f: B_{C_p}O(2) \rightarrow B_{C_p}\mathbb{Z}/2$  and then show that the Serre spectral sequence collapses with no extension problems.

Since  $\widetilde{B_{C_p}\mathbb{Z}/2}$  is the constant functor at  $S^\infty$ , the relevant equivariant Eilenberg spectral sequence in this case is

$$\mathrm{Ext}_{\Pi}^{u,v}(\mathcal{H}_*(S^\infty), h_{C_p}^{V+t}(f; A)) \Rightarrow H_{C_p}^{u+v}(B_{C_p}\mathbb{Z}/2; h_{C_p}^{V+t}(f; A)).$$

As before, in the  $E_2$  term,  $u$  is the homological degree and  $v$  is the internal grading on  $\mathcal{H}_*$ . Since  $\mathcal{H}_v(S^\infty)$  is either 0 or  $\mathbb{F}_q$ , both of which are projective,  $\mathrm{Hom}_{\Pi}(\mathcal{H}_v(S^\infty), -)$  is exact, and so all Ext terms with  $u > 0$  vanish. It follows that the spectral sequence collapses at  $E_2$  with no extension problems, and so the  $E_2$  terms of the Serre spectral sequence are given by

$$H_{C_p}^s(B_{C_p}\mathbb{Z}/2; h_{C_p}^{V+t}(f; A)) \cong \mathrm{Hom}_{\Pi}(\mathcal{H}_s(S^\infty), h_{C_p}^{V+t}(f; A)).$$

The homology of  $S^\infty$  vanishes for  $s > 0$ , so in fact the Serre spectral sequence also collapses with no extension problems.

$B_{C_p}\mathbb{Z}/2$  has a trivial  $C_p$  action and is  $C_p$ -connected, meaning that Proposition 3.3 and Proposition 3.4 apply. Thus we may identify

$$H_{C_p}^{V+t}(B_{C_p}O(2); A) \cong H_{C_p}^0(B_{C_p}\mathbb{Z}/2; h_{C_p}^{V+t}(f; A)) \hookrightarrow H_{C_p}^{V+t}(B_{C_p}SO(2); A)$$

as algebras over the cohomology of a point.

More specifically, we have

$$H_{C_p}^{V+t}(B_{C_p}O(2); A) \cong \mathrm{Hom}_{\Pi}(\mathbb{F}_q, h_{C_p}^{V+t}(f; A)).$$

Our coefficient systems take values in  $(\mathbb{F}_q)\text{-mod}$ , so for any  $\mathcal{N}$  and any element  $\eta \in \mathrm{Hom}_{\Pi}(\mathbb{F}_q, \mathcal{N})$ ,  $\eta$  factors through the “fixed subfunctor of  $\mathcal{N}$ ,” i.e. the subfunctor

$$\begin{array}{c} \mathcal{N}(C_p, x_0)^{\mathbb{Z}/2} \\ \downarrow \\ \mathcal{N}(\{e\}, x_0)^{C_p \times \mathbb{Z}/2}. \end{array}$$

Note the two downward arrows must give the same map, and so a single arrow has been drawn above. As already observed, our  $\Pi$  has a weakly terminal object, and so a map in  $\mathrm{Hom}_{\Pi}(\mathbb{F}_q, \mathcal{N})$  is in fact determined by choosing an element of  $\mathcal{N}(C_p, x_0)^{\mathbb{Z}/2}$ . By inspection of the structure of  $\Pi = \Pi_{C_p} B_{C_p}\mathbb{Z}/2$ , we see that every element of  $\mathcal{N}(C_p, x_0)^{\mathbb{Z}/2}$  defines a map in  $\mathrm{Hom}_{\Pi}(\mathbb{F}_q, \mathcal{N})$ , as well. In other words, we have demonstrated that

$$H_{C_p}^{V+t}(B_{C_p}O(2); A) \cong H_{C_p}^{V+t}(B_{C_p}SO(2); A)^{\mathbb{Z}/2}$$

for each  $V$  and  $t$ , and hence

$$H_{C_p}^*(B_{C_p}O(2); A) \cong H_{C_p}^*(B_{C_p}SO(2); A)^{\mathbb{Z}/2}.$$

By Theorem 6.3 and Proposition 7.5, it follows that  $H_{C_p}^*(B_{C_p}O(2); A)$  is the subalgebra of  $H_{C_p}^*(B_{C_p}SO(2); A)$  generated by the elements  $D_{2k}$ ,  $D_{2k-1}C$ , and  $C^2$ , the generators restricting to an even power of the nonequivariant generator  $x$  of  $H^*(\mathbb{C}P^\infty)$ .  $\square$

In fact, closer examination shows that the Green functor  $\underline{H}_{C_p}^*(B_{C_p}O(2))$  is a sub-Green functor of  $\underline{H}_{C_p}^*(CP(\mathcal{U})) = \underline{H}_{C_p}^*(B_{C_p}SO(2))$ , again on the generators  $D_{2k}$ ,  $D_{2k-1}C$ , and  $C^2$ .

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